Games with Sequential Actions: (Finite) Extensive-Form Games

Xinshuo Weng

Outline

- What are (finite) extensive-form (EF) games? (5.1.1 in the book)
  - Differences v.s. normal-form games, definition, perfect-information EF games
- Strategies and equilibria (5.1.2 in the book)
  - What are the strategies in perfect-information EF games and how to find the equilibria
- Subgame and subgame-perfect equilibrium (5.1.3 in the book)
  - What is a subgame and how to find subgame-perfect equilibrium

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1. What are (finite) extensive-form (EF) games?

- A **finite** representation that does not always assume that players act **simultaneously**.

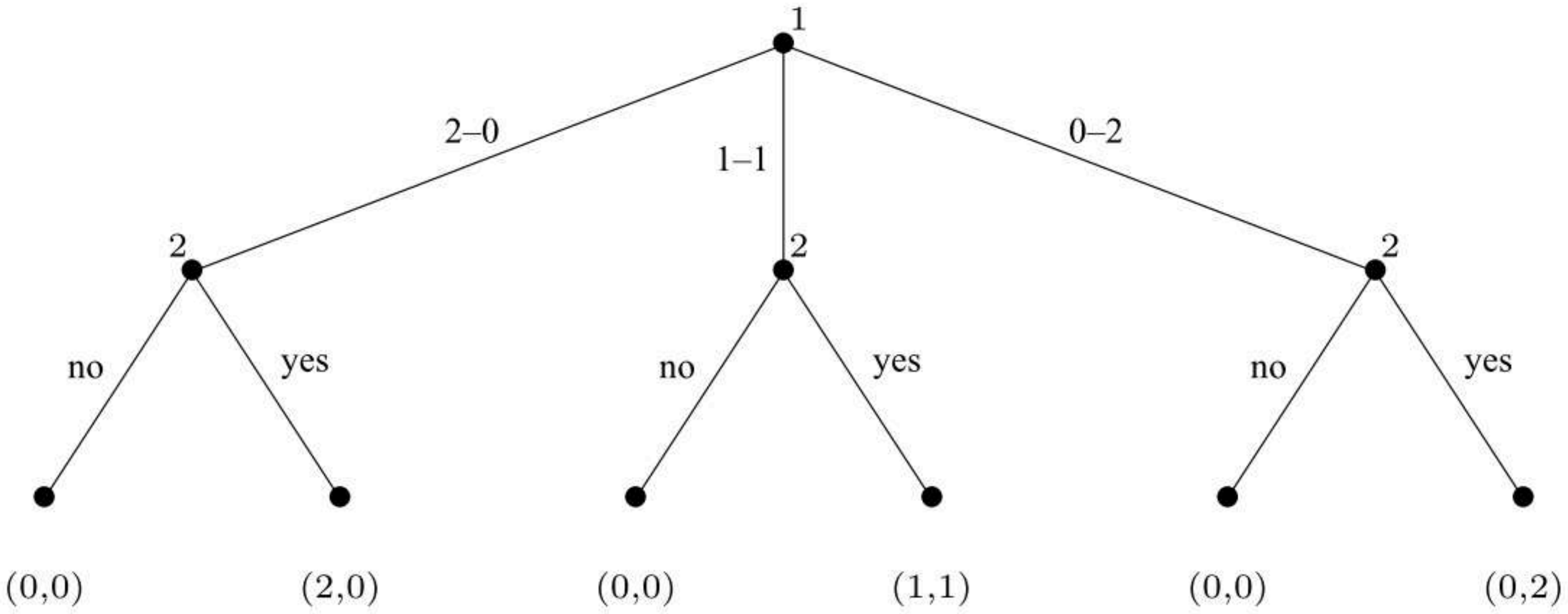


Figure 5.1: The Sharing game.

1. What are (finite) extensive-form (EF) games?

- A **finite** representation that does not always assume that players act **simultaneously**.
- Differences with respect to the normal-form game:
  - Tree v.s. table
  - Sequential play v.s. simultaneous play

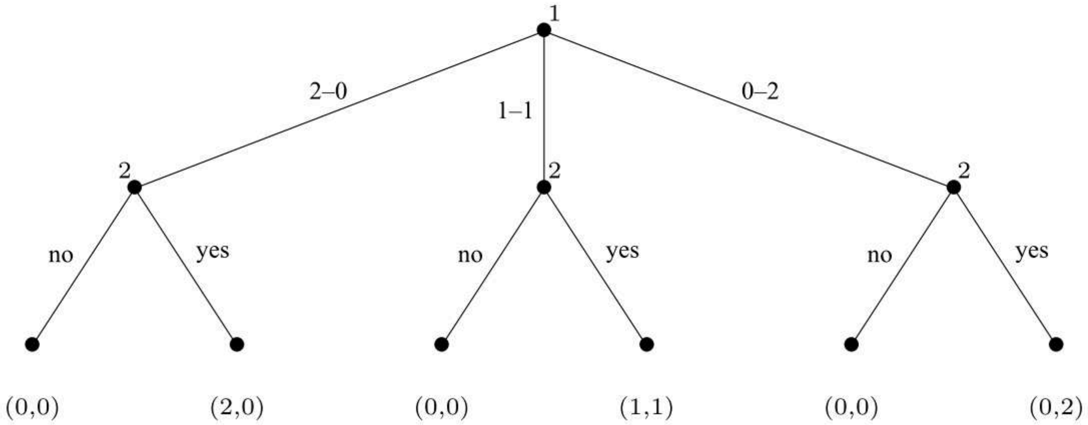


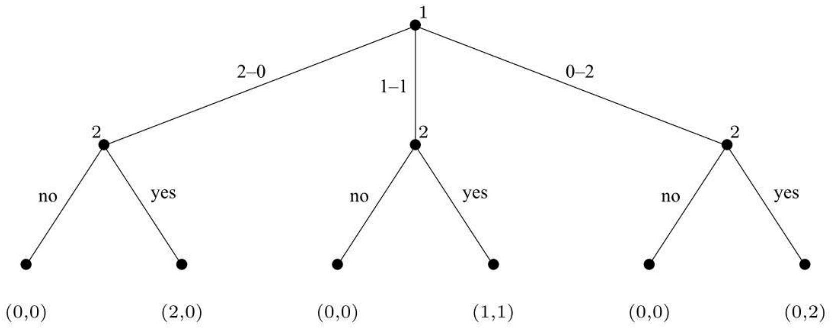
Figure 5.1: The Sharing game.

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Figure 3.6: Matching Pennies game.

1. What are (finite) extensive-form (EF) games?

- Informal definition:
    - (finite) Perfect-information EF games:
        - we allow players to specify the action that they would take at every node of the game. This implies that players know the node they are in (players **know what actions are played by other players**)



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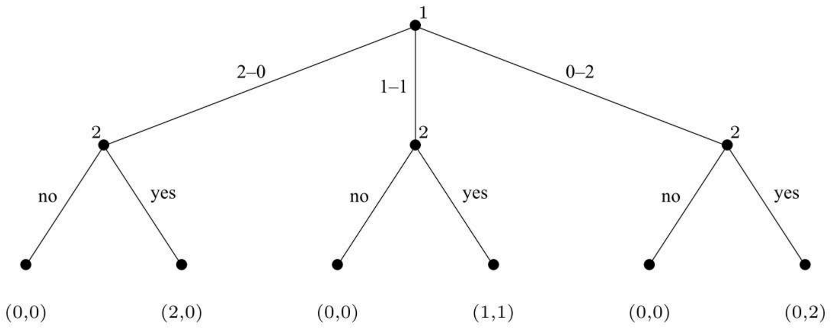
Perfect-information extensive-form

1. What are (finite) extensive-form (EF) games?

- Informal definition:
  - (finite) Perfect-information EF games:
    - players know the node they are in (players **know what actions are played by other players**)
  - (finite) Imperfect-information EF games:
    - each player's choice nodes are partitioned into information sets; intuitively, if two nodes are in the same information set then the agent cannot distinguish between them (**players do not know or partially know what actions are played by other players**)



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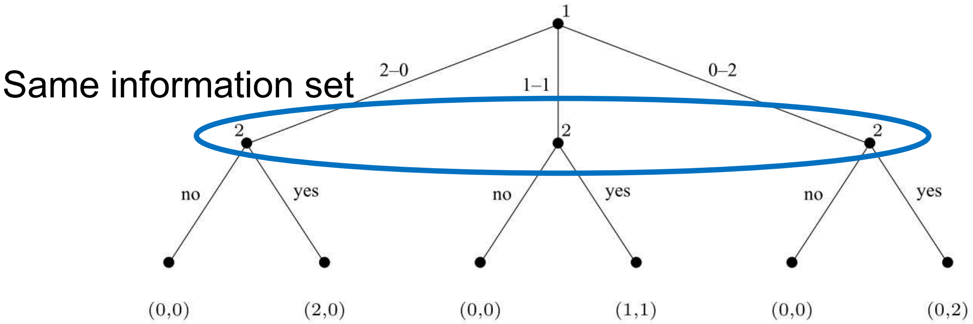
Perfect-information extensive-form



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Imperfect-information extensive-form

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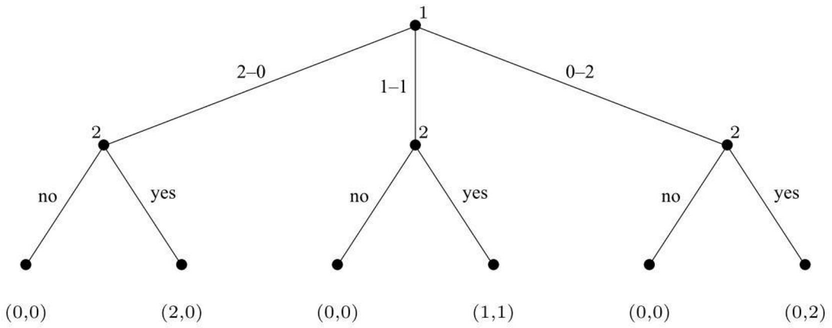

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Perfect-information extensive-form

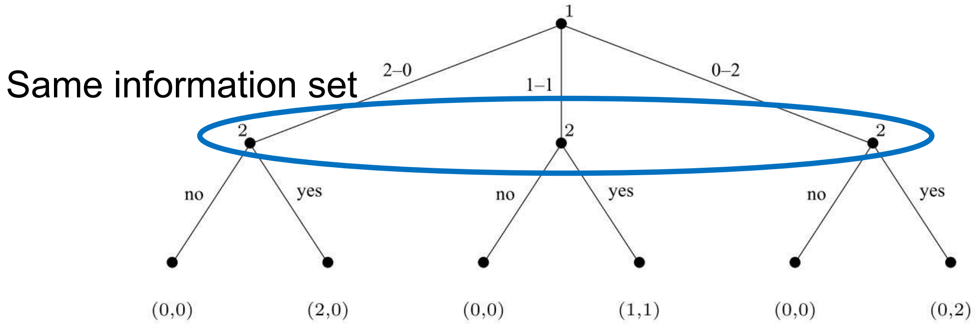

Figure 5.1: The Sharing game.

Imperfect-information extensive-form

1. What are (finite) extensive-form (EF) games?

Definition 5.1.1 (Perfect-information game) *A (finite) perfect-information game (in extensive form) is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where:*

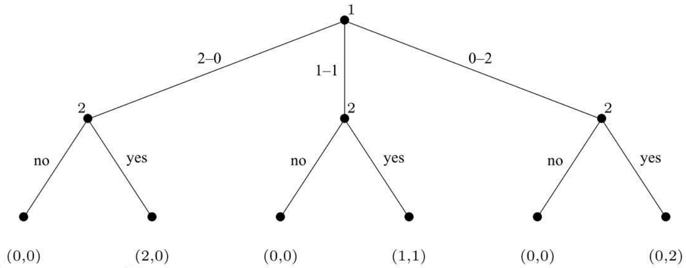

Figure 5.1: The Sharing game.

- *N is a set of n players;*
- *A is a (single) set of actions;*
- *H is a set of nonterminal choice nodes;*
- *Z is a set of terminal nodes, disjoint from H;*
- *$\chi : H \mapsto 2^A$ is the action function, which assigns to each choice node a set of possible actions;*
- *$\rho : H \mapsto N$ is the player function, which assigns to each nonterminal node a player $i \in N$ who chooses an action at that node;*
- *$\sigma : H \times A \mapsto H \cup Z$ is the successor function, which maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$; and*
- *$u = (u_1, \ldots, u_n)$, where $u_i : Z \mapsto \mathbb{R}$ is a real-valued utility function for player i on the terminal nodes Z.*

Outline

- What are (finite) extensive-form (EF) games? (5.1.1 in the book)
    - Differences v.s. normal-form games, definition, perfect-information EF games
- **Strategies and equilibria (5.1.2 in the book)**
    - What are the strategies in perfect-information EF games and how to find the equilibria
- Subgame and subgame-perfect equilibrium (5.1.3 in the book)
    - What is a subgame and how to find subgame-perfect equilibrium

2. Strategies and equilibria

Definition 5.1.2 (Pure strategies) *Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consist of the Cartesian product $\prod_{h \in H, \rho(h)=i} \chi(h)$.*

Pure strategies for player i is the product of the set of possible actions at all choice nodes that player i needs to take action

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Exercise

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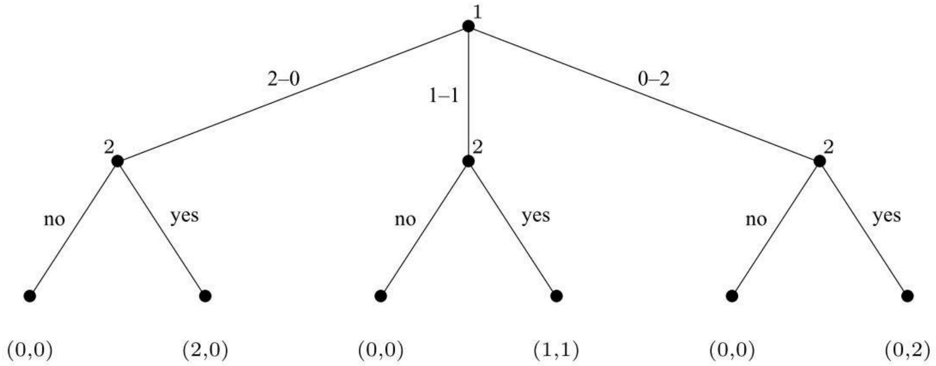
2. What are the pure strategies S for play 2?



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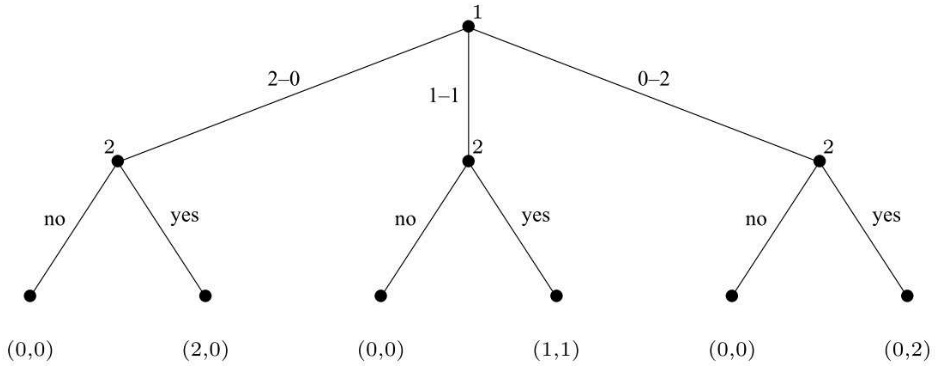

Figure 5.1: The Sharing game.

$S_1 = \{2\text{–}0, 1\text{–}1, 0\text{–}2\}$

$S_2 = \{(yes, yes, yes), (yes, yes, no), (yes, no, yes), (yes, no, no), (no, yes, yes), (no, yes, no), (no, no, yes), (no, no, no)\}$

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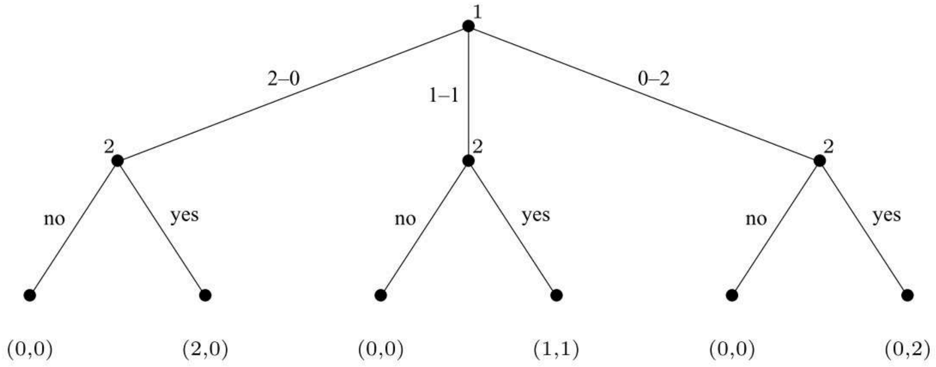
Note: "an agent's strategy requires a decision at each choice node, regardless of whether or not it is possible to reach that node given the other choice nodes (before the game starts)"



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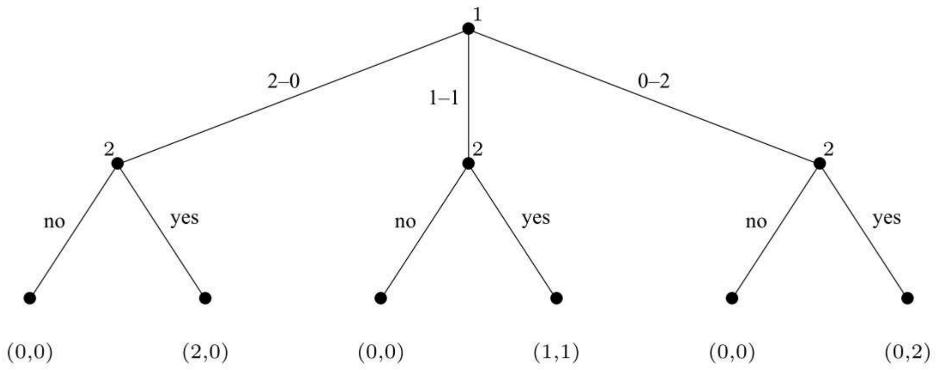
For the sharing game, it is possible to reach every node for each player
Thus no confusion for finding the pure strategies



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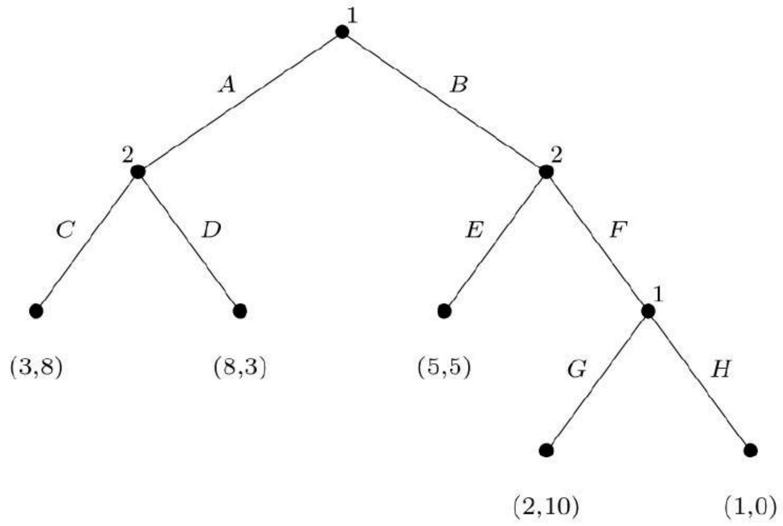

Figure 5.2: A perfect-information game in extensive form.

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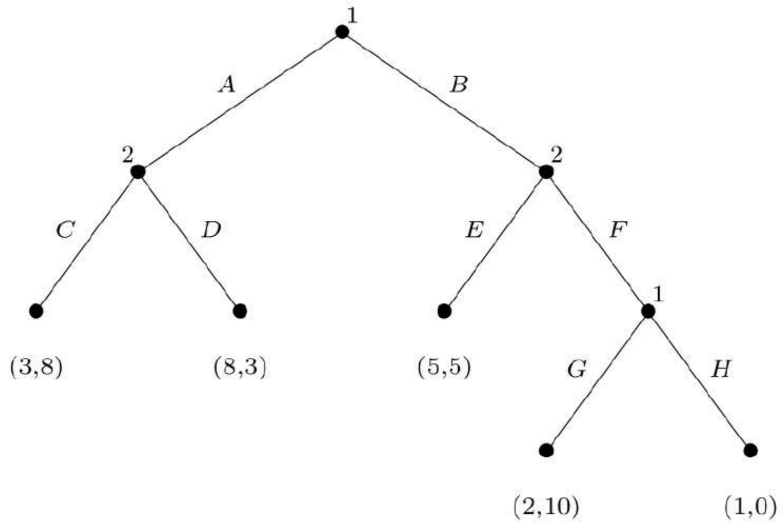
$$S_1 = \{(A, G), (A, H), (B, G), (B, H)\}$$

$$S_2 = \{(C, E), (C, F), (D, E), (D, F)\}$$



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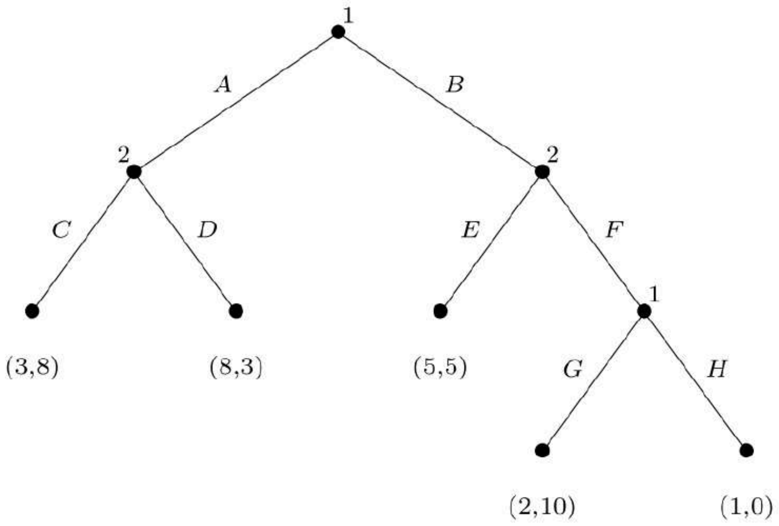
Need to include these two strategies even though node G and H are not reachable



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2. Strategies and equilibria

Theorem 5.1.3 *Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium.*

"The intuition is that, since players take turns, and everyone gets to see everything that happened thus far before making a move, it is never necessary to introduce randomness into action selection in order to find an equilibrium."

Mixed-strategy and its Nash equilibrium will be discussed in imperfect-information extensive-form game

2. Strategies and equilibria

"For ***every*** *perfect-information extensive-form game, there exists a corresponding normal-form game, called 'induced normal-form game', which preserves game-theoretic properties such as Nash equilibria."*

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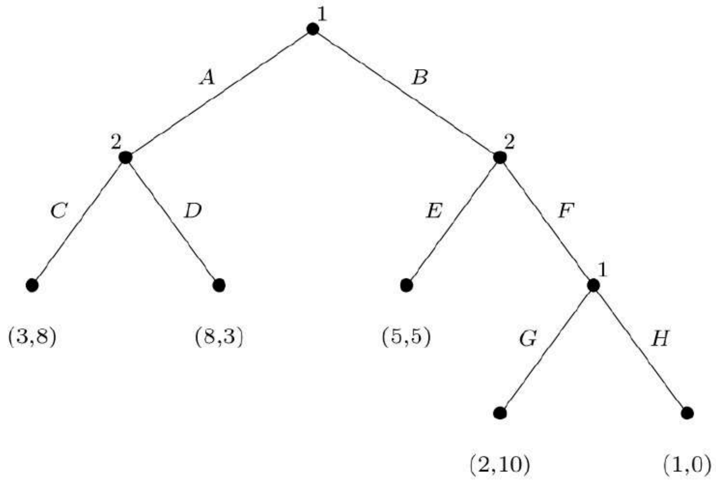

Figure 5.2: A perfect-information game in extensive form.

$$S_1 = \{(A, G), (A, H), (B, G), (B, H)\}$$

$$S_2 = \{(C, E), (C, F), (D, E), (D, F)\}$$

The original game

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

Figure 5.3: The game from Figure 5.2 in normal form.

Induced normal-form game

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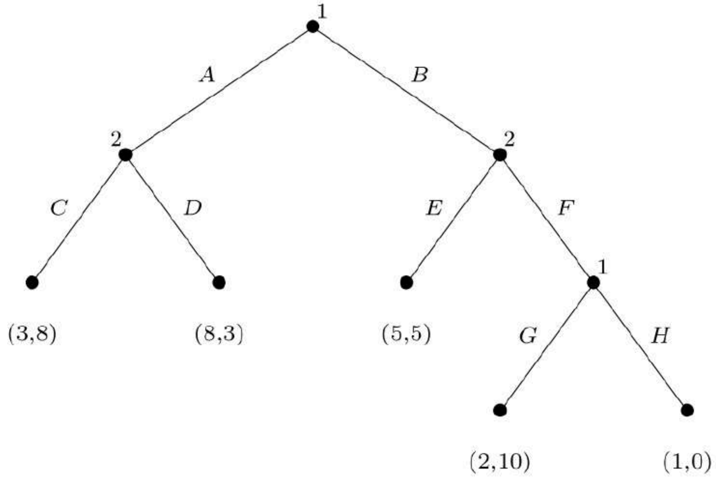

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$$S_1 = \{(A,G),(A,H),(B,G),(B,H)\}$$

$$S_2 = \{(C,E),(C,F),(D,E),(D,F)\}$$

The original game

Nash Equilibria

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
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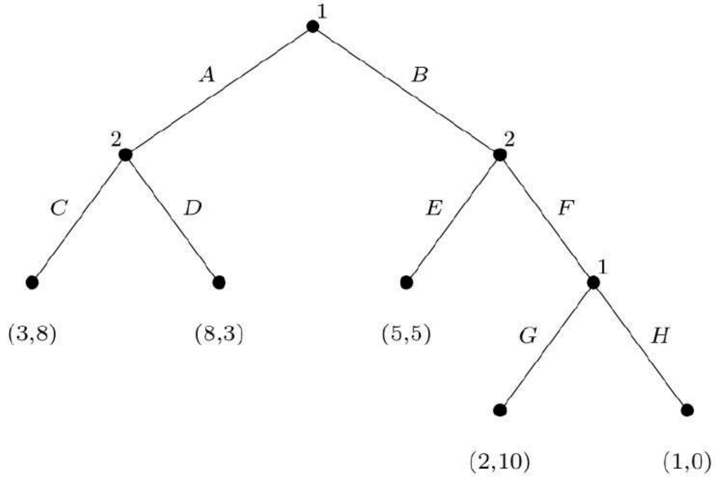
Induced normal-form game

2. Strategies and equilibria

Disadvantages of the induced normal-form game:

- *Redundancy, can result in an exponential blowup of the game representation*
- *Lose the temporal structure*



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The reverse transformation does not always exist because the perfect-information extensive-form game cannot model the simultaneous move by all players
e.g., matching pennies game

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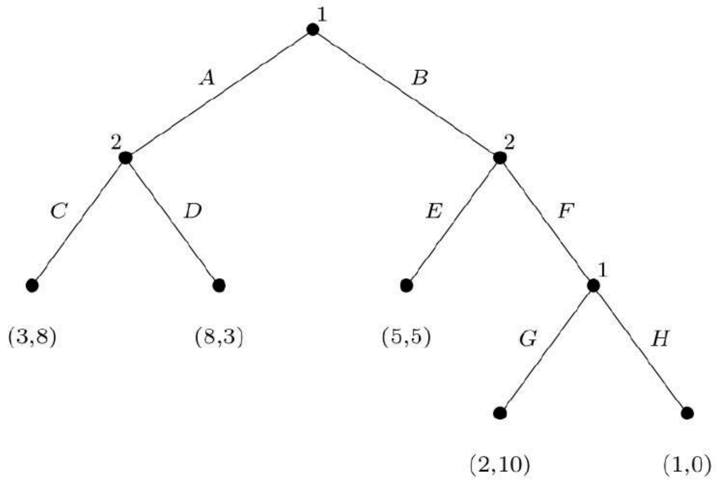
3. Subgame and subgame-perfect equilibrium

Are all the Nash equilibrium satisfying in the extensive-form game?

No. Considering (B,H) (C,E)...



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The original game

Nash Equilibria

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(A,H)	3, 8	3, 8	8, 3	8, 3
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Induced normal-form game

3. Subgame and subgame-perfect equilibrium

Are all the Nash equilibrium satisfying in the extensive-form game?

No. Considering (B,H) (C,E), which is locally unsatisfying and contains noncredible threats

Nash Equilibria

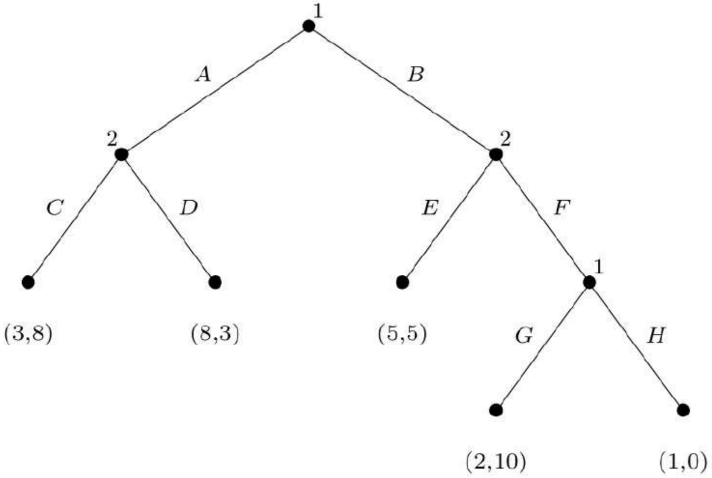

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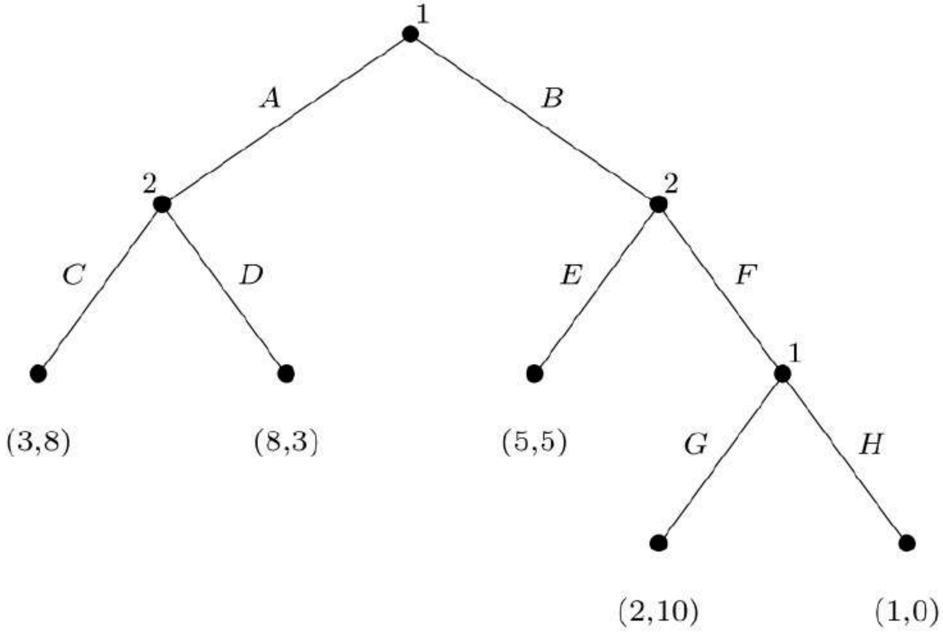
Figure 5.3: The game from Figure 5.2 in normal form.

Induced normal-form game

3. Subgame and subgame-perfect equilibrium

Definition 5.1.4 (Subgame) *Given a perfect-information extensive-form game G, the* subgame *of G rooted at node h is the restriction of G to the descendants of h. The set of subgames of G consists of all of subgames of G rooted at some node in G.*



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Definition 5.1.5 (Subgame-perfect equilibrium) *The* subgame-perfect equilibria *(SPE) of a game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.*

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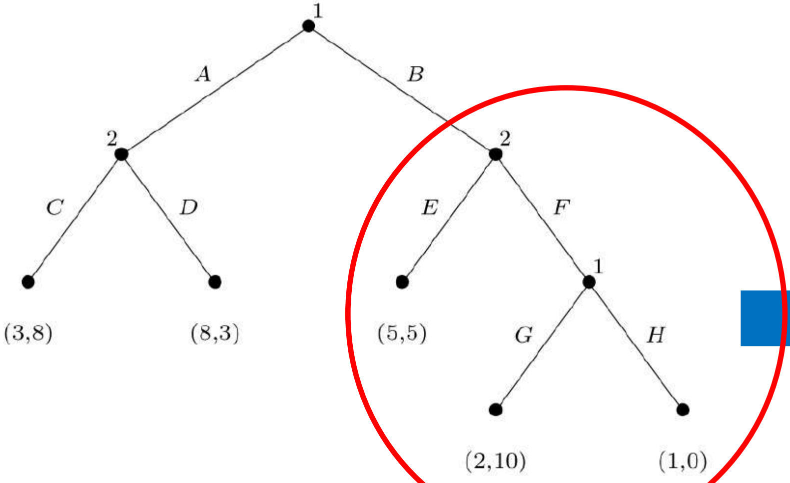

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The original game

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(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
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(B,H)	5, 5	1, 0	5, 5	1, 0

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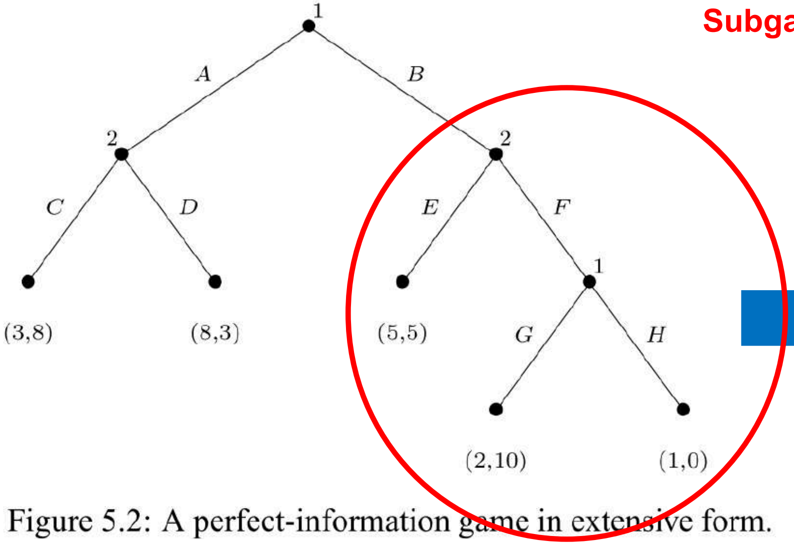

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The original game

Nash Equilibria

Subgame-Perfect Equilibria

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SPE is a stronger concept than Nash equilibrium (i.e., every SPE is a NE, but not every NE is a SPE

SPE can rule out "noncredible threats" that might exist in NE

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